Rineke Verbrugge, manuscript, 1992

Verzamelingen-Veltman frames en modellen

Definitie 1.

Een *IL_{Set}frame* is een L-frame $\langle W,R\rangle$ met een extra relatie S_w voor iedere $w\in W$, met de volgende eigenschappen:

(I) $S_w \subseteq W[w] \times P(W[w]) \setminus \emptyset$

(II) S_w is "reflexief": Als wRx, dan $xS_w\{x\}$

S_w is "transitief": Als xS_wY dan geldt voor alle $y\in Y$ en alle $V\in P(W[w])$: $yS_wV \Rightarrow xS_wV$.

(III) Als $wRw'Rw''$, dan $w'S_w\{w''\}$

English translation on page 1a

Een *IL_{Set} model* bestaat uit een IL_{Set} frame $\langle W,R,S\rangle$ met een forcing relatie $\Vdash$ die voldoet aan:

$$u\Vdash\Box\varphi \iff \forall v(uRv \Rightarrow v\Vdash\varphi)$$

$$u\Vdash\varphi\rhd\psi \iff \forall v(uRv \text{ en } v\Vdash\varphi \Rightarrow \exists V(vS_uV \text{ en } \forall \overset{x}{\cancel{y}}\in V\ \overset{x}{\cancel{y}}\Vdash\psi))$$

Het is niet moeilijk in te zien dat voor ieder IL_{Set} model K, $K\models IL$.

From IL_{Set} models to IL models

Theorem 2.

Let $\langle W,R,S,\Vdash\rangle$ be an IL_{Set} model. Then there is an IL model $\langle W',R',S',\Vdash'\rangle$ and a bijection $f:W\rightarrow P(W')$ such that for all $w\in W$ and all $w'\in f(w)$: $w\Vdash\varphi \iff w'\Vdash'\varphi$.

Proof.

Let W' consist of all points $\langle x,A\rangle$, where $x\in W$ and A is a set of ordered pairs such that:

For all w, V with xS_wV, there is a $v\in V$ with $\langle w,v\rangle\in A$;

and, conversely, if $\langle w,v\rangle\in A$, then there is a V such that $v\in V$ and xS_wV.

Define R', S' as follows:

$\langle x,A\rangle R'\langle y,B\rangle \iff xRy$ and for all w such that wRx and all z:
if $\langle w,z\rangle\in B$, then $\langle w,z\rangle\in A$.

$\langle x,A\rangle S'_{\langle w,C\rangle}\langle y,B\rangle \iff \langle w,C\rangle R'\langle x,A\rangle$, $\langle w,C\rangle R'\langle y,B\rangle$ and for all v:

if $\langle w,v\rangle \in B$, then $\langle w,v\rangle \in A$
(thus in particular, because S_w's being "reflexive" implies $\langle w,y\rangle \in B$, we have $\langle w,y\rangle \in A$).

Finally, define $\Vdash'$ as follows:
$\langle x,A\rangle \Vdash' p \iff x \Vdash p$.

We will prove that
(a) $\langle W',R',S'\rangle$ is an IL frame
(b) For all φ, $\langle x,A\rangle \Vdash' \varphi \iff x \Vdash \varphi$

Proof of (a):
First of all, it is not difficult to see that $\langle W',R'\rangle$ is an L-frame. Checking the clauses for $S'_{\langle w,C\rangle}$ requires a bit more work.

(I') If $\langle w,C\rangle \in W'$, then $S'_{\langle w,C\rangle}$ is a relation on $W'[\langle w,C\rangle]$; this follows immediately from the definition of $S'_{\langle w,C\rangle}$.

(II') $S'_{\langle w,C\rangle}$ is reflexive; for suppose $\langle w,C\rangle R' \langle x,A\rangle$, then by the definition of W': $\langle x,A\rangle S'_{\langle w,C\rangle} \langle x,A\rangle$.
$S'_{\langle w,C\rangle}$ is transitive; for suppose $\langle x,A\rangle S'_{\langle w,C\rangle} \langle y,B\rangle S'_{\langle w,C\rangle} \langle z,D\rangle$, then $\langle w,C\rangle R' \langle x,A\rangle$, $\langle w,C\rangle R' \langle z,D\rangle$ and for all v : if $\langle w,v\rangle \in D$, then $\langle w,v\rangle \in B$, and thus $\langle w,v\rangle \in A$; therefore $\langle x,A\rangle S'_{\langle w,C\rangle} \langle z,D\rangle$.

(III') If $\langle w,C\rangle R' \langle x,A\rangle R' \langle y,B\rangle$, then wRx and by definition of R': for all z, if $\langle w,z\rangle \in B$, then $\langle w,z\rangle \in A$; therefore $\langle x,A\rangle S'_{\langle w,C\rangle} \langle y,B\rangle$.

Proof of (b):
As usual, the only interesting case is the induction step for $\rhd$.

Suppose $w \Vdash \varphi \rhd \psi$ and $\langle w,C\rangle \in W'$. We want to prove $\langle w,C\rangle \Vdash' \varphi \rhd \psi$. So suppose $\langle w,C\rangle R' \langle x,A\rangle$ and $\langle x,A\rangle \Vdash' \varphi$.
Then wRx and, by the induction hypothesis, $x \Vdash \varphi$. Therefore, there is a V with xS_wV and $\forall y \in V\ y \Vdash \psi$. We want to find a B such that $\langle x,A\rangle S'_{\langle w,C\rangle} \langle y,B\rangle$. This is possible because of the following two facts:

(1) By "transitivity" of S_w, we have $\forall V\ (yS_wV \Rightarrow xS_wV)$

(2) For any b with bRwRy we have, by (III), $wS_b\{y\}$, and thus by transitivity: if yS_bV, then wS_bV.

Therefore, we can take a B such that $\langle y,B\rangle \in W'$ and, by (1), for all v, if $\langle w,v\rangle \in B$, then $\langle w,v\rangle \in A$; moreover wRy and, by (2), if bRw, then for all z: if $\langle b,z\rangle \in B$, then $\langle b,z\rangle \in C$, so $\langle w,C\rangle R'\langle y,B\rangle$. In conclusion, $\langle x,A\rangle S'_{\langle w,C\rangle}\langle y,B\rangle$ and by induction hypothesis, $\langle y,B\rangle \Vdash' \psi$.

Suppose on the other hand that $w \Vdash \neg(\varphi \rhd \psi)$. We will prove $\langle w,C\rangle \Vdash' \neg(\varphi \rhd \psi)$.

First, $w \Vdash \neg(\varphi \rhd \psi)$ implies that there is an x with wRx and for all V with xS_wV there is a $y \in V$ such that $y \Vdash \neg\psi$.

Therefore, it is possible to take an A such that $\langle x,A\rangle \in W'$ and for all y: if $\langle w,y\rangle \in A$, then $y \Vdash \neg\psi$, and moreover $\langle w,C\rangle R'\langle x,A\rangle$ (the extra clause for R' doesn't interfere with our desiderations for A).

For this $\langle x,A\rangle$, we have by induction hypothesis $\langle x,A\rangle \Vdash \varphi$.

Moreover, if $\langle x,A\rangle S'_{\langle w,C\rangle}\langle y,B\rangle$, then $\langle w,y\rangle \in B$, and thus $\langle w,y\rangle \in A$ [see earlier remark], so by induction hypothesis $\langle y,B\rangle \Vdash \neg\psi$

<u>Def</u> $W[w] := \{w' \in W \mid wRw'\}$

<u>Def</u>

An <u>IL_{Set} frame</u> is an L-frame $\langle W, R\rangle$ + for each $w \in W$, a relation S_w with the following properties:

(I) $S_w \subseteq W[w] \times \mathcal{P}(W[w]) \setminus \{\emptyset\}$

(II) S_w is "quasi-reflexive":
for all w, x, $\quad wRx \longrightarrow x S_w \{x\}$

S_w is "quasi-transitive":
$xS_w Y \longrightarrow \forall y \in Y, \forall V \in \mathcal{P}(W[w])(y S_w V \Rightarrow x S_w V)$

(III) $wRw'Rw'' \longrightarrow w' S_w \{w''\}$.

An <u>IL_{Set} model</u> consists of an IL_{Set}-frame $\langle W, R, \{S_w : w \in W\}\rangle$ + a forcing relation $\Vdash$ satisfying the following:

$$u \Vdash \Box\varphi \iff \forall v\,(uRv \Rightarrow v \Vdash \varphi)$$
$$u \Vdash \varphi \rhd \psi \iff \forall v\,(uRv \wedge v \Vdash \varphi \Rightarrow \exists V(v S_u V \wedge \forall x \in V\ x \Vdash \psi$$

It is not difficult to check that for any IL_{Set}-model K, $K \models IL$.

It seems that for some applications, transitivity is more nicely defined as:
$$x S_w Y \wedge \forall y \in Y (y S_w Z_y) \rightarrow x S_w \bigcup_{y \in Y} Z_y$$
+ extra condition:
$$x S_w Y \wedge Y \subseteq Z \longrightarrow x S_w Z.$$
We didn't check what happens in this case, and in the rest of these pages we always use the first definition

$M: A \rhd B \longrightarrow A \wedge \Box C \rhd B \wedge \Box C$

What property does M characterize on set models?

$M*)$ $u S_w V \longrightarrow \exists V' \subseteq V, (u S_w V' \wedge \forall v \in V'(vRz \to uRz))$

Proof

1) Suppose our frame satisfies $M*$ and $w \Vdash A \rhd B$.

Suppose wRu, $u \Vdash A \wedge \Box C$. Because $w \Vdash A \rhd B$,

$\exists V$ $u S_w V$ & $\forall v \in V$ $v \Vdash B$.

Now by $*$,

$\exists V' \subseteq V$ $(u S_w V' \wedge \forall v \in V'(vRz \to uRz))$

so $\forall v \in V'$, $v \Vdash B \wedge \Box C$.

Therefore $w \Vdash A \wedge \Box C \rhd B \wedge \Box C$.

2) Suppose our frame does not satisfy $M*$, i.e.

suppose $u S_w V$ but $\forall V' \subseteq V(u S_w V' \to \exists v \in V'(vRz \wedge \neg uRz$

now take a valuation which:

- forces p only in u
- forces r in V but nowhere else
- Does not force q in those $v \in V$ with $vRz \wedge \neg uRz$, but does force q everywhere else.

Then $w \not\Vdash p \wedge \Box q \rhd r \wedge \Box q$,

because wRu, $u \Vdash p \wedge \Box q$.

But for any V' with $u S_w V'$ and $V' \Vdash r$, we have $V' \subseteq V$, so $V' \not\Vdash \Box q$.

= KM1

Π: $A \rhd \Diamond B \longrightarrow \Box(A \to \Diamond B)$

What property is characterized by Π on set frames
(On normal frames it is the same as the M-property)

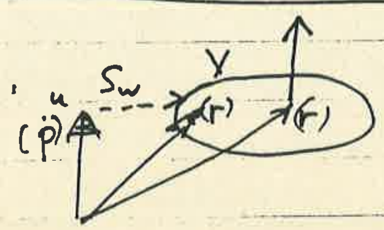
$\Pi *$: If $u S_w V$, then $\exists v \in V \, \forall z (vRz \to uRz)$

Proof 1) Suppose our frame satisfies $\Pi *$,
and suppose $w \Vdash A \rhd \Diamond B$. Suppose wRu and $u \Vdash A$
Then there is a V such that $u S_w V$, and
for all $v \in V \, \exists z_v \; vRz_v$ & $z_v \Vdash B$.
By $*$), $\exists v \in V$ such that $\forall z (vRz \to uRz)$,
So, especially uRz_v, thus $u \Vdash \Diamond B$.
Therefore $w \Vdash \Box(A \to \Diamond B)$.

2) Suppose our frame does not satisfy $\Pi *$, i.e. we suppose
$u S_w V$ & $\forall v \in V \, \exists z_v \, (vRz_v \wedge \neg uRz_v)$.
We take a valuation such that
- p is forced only in u
- q is forced only in the "z_v's".

Then $w \Vdash p \rhd \Diamond q$, because
$u S_w V$ and $\forall v \in V$, $v \Vdash \Diamond q$ (because $z_v \Vdash q$).
But not $w \Vdash \Box(p \to \Diamond q)$, because wRu and
$u \nVdash \Diamond q$ (u has no R-arrow to any "z_v").

ILΠ $\nvDash$ M :



Countermodel

This frame satisfies property $\Pi *$, but
$w \nVdash p \rhd r \longrightarrow p \wedge \Box\bot \rhd r \wedge \Box\bot$

$\underline{P : A \rhd B \to \Box(A \rhd B)}$

What property does P characterize on set frames?

$*P$: $u S_w V \wedge w R w' R u \to \exists V' \subseteq V \; u S_{w'} V'$.

Proof

1) Suppose our frame satisfies $*P$,
and suppose $w \Vdash A \rhd B$.
Moreover suppose wRw', $w'Ru$ and $u \Vdash A$.
We have by transitivity wRu, so there is a V
with $u S_w V$ and $\forall v \in V \; v \Vdash B$.
By $*P$, $\exists V' \subseteq V \; u S_{w'} V'$, and $\forall v \in V' \; v \Vdash B$.
So $w' \Vdash A \rhd B$, thus $w \Vdash \Box(A \rhd B)$.

2) Suppose our frame does not satisfy $*P$, e.g.
suppose
$u S_w V$, $wRw'Ru$, but $\forall V' \subseteq V$ not $u S_{w'} V'$.
Take a valuation such that
- p is forced only in u
- q is forced everywhere in V, but nowhere else.

Then $w \Vdash p \rhd q$, but
$w' \nVdash p \rhd q$, so $w \nVdash (p \rhd q) \to \Box(p \rhd q)$.

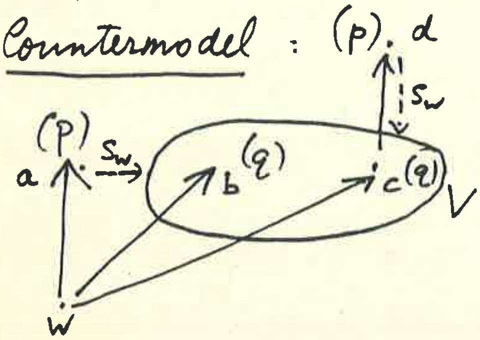
F : $\varphi \rhd \Diamond\varphi \longrightarrow \Box\neg\varphi$

W : $\varphi \rhd \psi \longrightarrow \varphi \rhd \psi \wedge \Box\neg\varphi$

KW1 : $\varphi \rhd \Diamond\top \longrightarrow \top \rhd \neg\varphi$

<u>Th</u> $ILF \nvDash W$

<u>Countermodel</u> :



1) $w \nVdash p \rhd q \longrightarrow p \rhd q \wedge \Box\neg p$.

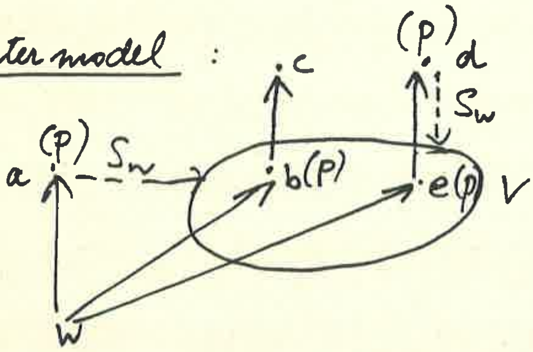
<u>Pf</u>. For, $w \Vdash p \rhd q$: from both points in which p holds (a and d), one can reach V by S_w, and $V \Vdash q$.
Also $w \nVdash p \rhd q \wedge \Box\neg p$: from a, we cannot reach any V' with $V' \Vdash q \wedge \Box\neg p$; because q holds only in V, and $V \nVdash \Box\neg p$ because cRd, $d \Vdash p$.

2) for all φ, $w \Vdash \varphi \rhd \Diamond\varphi \longrightarrow \Box\neg\varphi$. So, $w \Vdash ILF$

<u>Pf</u>. Notice first that a and d are indistinguishable: They force exactly the same sentences (proof by induction).
Now suppose that $w \Vdash \varphi \rhd \Diamond\varphi$.
Suppose $a \Vdash \varphi$, Then we must have $a \Vdash \Diamond\varphi$ or $V \Vdash \Diamond\varphi$ $\lightning$
So $a \nVdash \varphi$ and $d \nVdash \varphi$.
Suppose $c \Vdash \varphi$, Then $c \Vdash \Diamond\varphi$ and $d \Vdash \varphi$ $\lightning$, so $c \nVdash \varphi$.
Suppose $b \Vdash \varphi$, then $b \Vdash \Diamond\varphi$ $\lightning$, so $b \nVdash \varphi$.
Thus, $w \Vdash \Box\neg\varphi$.

<u>Th</u> ILF $\nvDash$ KW1

<u>Countermodel</u> :



1) $w \nVdash p \rhd \Diamond\top \longrightarrow \top \rhd \neg p$

<u>Pf</u> $w \Vdash p \rhd \Diamond\top$: from a, we can reach by S_w V, and $V \Vdash \Diamond\top$

from b, we reach $\{b\}$, and $b \Vdash \Diamond\top$

from e, we reach $\{e\}$ and $e \Vdash \Diamond\top$

from d, we reach V , and $V \Vdash \Diamond\top$.

$w \nVdash \top \rhd \neg p$: from a, we can not reach by S_w any set V' with $V' \Vdash \neg p$.

2) For all φ, $w \Vdash \varphi \rhd \Diamond\varphi \to \Box\neg\varphi$.

<u>Pf</u>. Again, a and d are indistinguishable.

Suppose that $w \Vdash \varphi \rhd \Diamond\varphi$. Suppose $c \Vdash \varphi$. Then $c \Vdash \Diamond\varphi$ $\lightning$. : $c \nVdash$

Suppose $a \Vdash \varphi$, then either $a \Vdash \Diamond\varphi$ or $V \Vdash \Diamond\varphi$.

so especially $c \Vdash \varphi$ $\lightning$. So $a \nVdash \varphi$ and $d \nVdash \varphi$.

Suppose $b \Vdash \varphi$, Then either $b \Vdash \Diamond\varphi$ or $c \Vdash \varphi$, in both cases $c \Vdash \varphi$ $\lightning$, so $b \nVdash \varphi$

Suppose $e \Vdash \varphi$, then $e \Vdash \Diamond\varphi$ or $d \Vdash \varphi$, in either case $d \Vdash \varphi$ $\lightning$, so $e \nVdash \varphi$.

We conclude $w \nVdash \Box\varphi$