Zero-one laws with respect to models of provability logic and two Grzegorczyk logics

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1 Introduction

In the late 1960s, Glebskii and colleagues proved that first-order logic without function symbols satisfies a zero-one law: every formula is either almost always true or almost always false in finite models [6]. More formally, let L be a language of first-order logic and let $A_n(L)$ be the set of all L-models with universe $\{1, \ldots, n\}$. Now let $\mu_n(\sigma)$ be the fraction of members of $A_n(L)$ in which σ is true:

$$\mu_n(\sigma) = \frac{\mid M \in A_n(L) : M \models \sigma \mid}{\mid A_n(L) \mid}$$

Then for every $\sigma \in L$, $\lim_{n \to \infty} \mu_n(\sigma) = 1$ or $\lim_{n \to \infty} \mu_n(\varphi) = 0$. This was also proved later but independently by Fagin [5]; Carnap had already proved the zero-one law for first-order languages with only unary predicate symbols [3].

The above zero-one laws and other limit laws have found applications in database theory and AI.In this article, we are interested in zero-one laws for some modal logics that impose structural restrictions on their models; all three logics that we are interested in are sound and complete with respect to finite partial orders, with different extra restrictions per logic. The zero-one law for first-order logic also holds when restricted to partial orders, both reflexive and irreflexive ones [4]. The proof uses a surprising combinatorial result by Kleitman and Rothschild [9] on which we will also rely for our results.

1.1 Kleitman and Rothschild's result on finite partial orders

Kleitman and Rothschild proved that with asymptotic probability 1, finite partial orders have a very special structure: There are no chains $u < v < w < z$ of more than three objects and the structure can be divided into three levels:

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- L_1, the set of minimal elements;
- L_2, the set of elements immediately succeeding elements in L_1;
- L_3, the set of elements immediately succeeding elements in L_2.

Moreover, in partial orders of size n, the sizes of these sets tend to $\frac{n}{4}$ for both L_1 and L_3 while the size of L_2 tends to $\frac{n}{2}$. As n increases, each element in L_1 has as immediate successors asymptotically half of the elements of L_2 and each element in L_2 has as immediate successors asymptotically half of the elements of L_3 [9]. Kleitman and Rothschild's theorem holds both for reflexive (non-strict) and for irreflexive (strict) partial orders.

1.2   Zero-one laws for modal logics

Let $\Phi = \{p_1, \ldots, p_k\}$ be a finite set of propositional atoms [2] and let $L(\Phi)$ be the modal language over Φ, inductively defined as the smallest set closed under:

(i) If $p \in \Phi$, then $p \in L(\Phi)$.

(ii) If $A \in L(\Phi)$ and $B \in L(\Phi)$, then also $\neg A \in L(\Phi)$, $\Box A \in L(\Phi)$, $\Diamond(\varphi) \in L(\Phi)$, $(A \wedge B) \in L(\Phi)$, $(A \vee B) \in L(\Phi)$, and $(A \rightarrow B) \in L(\Phi)$.

Let $\mathcal{M}_{n,\Phi}$ be the set of finite Kripke models over Φ with set of worlds $W = \{1, \ldots, n\}$. We take $\nu_{n,\Phi}$ to be the uniform probability distribution on $\mathcal{M}_{n,\Phi}$. Let $\nu_{n,\Phi}(\varphi)$ be the measure in $\mathcal{M}_{n,\Phi}$ of the set of Kripke models in which φ is valid. Halpern and Kapron proved that every formula φ in $L(\Phi)$ is either valid in almost all models or not valid in almost all models [8, Corollary 4.2]:

$$\text{Either} \lim_{n \to \infty} \nu_{n,\Phi}(\varphi) = 0 \text{ or } \lim_{n \to \infty} \nu_{n,\Phi}(\varphi) = 1.$$

By the Kleitman-Rothschild theorem, this modal zero-one law can also be restricted to finite models on reflexive or irreflexive partial orders, so that the existence of zero-one laws for finite models of provability logic and Grzegorczyk logic immediately follow. However, one would like to prove a stronger result and axiomatize the set formulas φ for which $\lim_{n \to \infty} \nu_{n,\Phi}(\varphi) = 1$.

The result about **GL** was proved in my 1995 LMPS presentation [12], but the proof was not published before. The 0-1 laws for **Grz** and **WGrz** are new.

2   Provability logic and two of its cousins

Here follow brief reminders about provability logic **GL**, Grzegorczyk logic **Grz**, and weak Grzegorczyk logic **wGrz**.

2.1   Provability Logic

The most widely used provability logic is called **GL** after Gödel and Löb. As axioms, it contains all axiom schemes from **K** and the extra scheme GL:

$$\text{All (instances of) propositional tautologies} \qquad \text{(A1)}$$
$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) \qquad \text{(A2)}$$
$$\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi \qquad \text{(GL)}$$

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[2] In the rest of this paper, we take Φ to be finite, although the results can be extended to enumerably infinite Φ by the methods described in [8].

The rules of inference of **GL** are modus ponens and necessitation (if $\mathbf{GL} \vdash \varphi$, then $\mathbf{GL} \vdash \Box\varphi$). Note that $\mathbf{GL} \vdash \Box\varphi \to \Box\Box\varphi$, as first proved by De Jongh and Sambin [1,13], but that the reflexivity axiom $\Box\varphi \to \varphi$ does not follow. Indeed, Segerberg proved in 1971 that provability logic is sound and complete with respect to all finite, transitive, irreflexive frames [11].

2.2 Grzegorczyk logic

Grzegorczyk Logic **Grz**, first introduced in [7], has the same axiom schemes and inference rules as **GL**, except that axiom GL is replaced by Grz:
$$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi \tag{Grz}$$

Grz is sound and complete with respect to the class of all finite transitive, reflexive and anti-symmetric frames [1, Chapter 12].

2.3 Weak Grzegorczyk logic

Weak Grzegorczyk Logic **Grz** has the same axiom schemes and inference rules as **GL**, except that axiom GL is replaced by wGrz, in which $\Box^+\psi := \Box\psi \wedge \psi$:
$$\Box^+(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi \tag{wGrz}$$

wGrz is sound and complete with respect to the class of all finite transitive, anti-symmetric frames (which need be neither irreflexive nor reflexive) [10].

3 Zero-one laws over relevant classes of finite models

3.1 GL: 0-1 law in finite irreflexive transitive models

We provide an axiomatization for almost sure model validity with respect to the relevant finite models corresponding to **GL**, namely the irreflexive transitive ones. The axiom system $\mathbf{AX}_{\mathbf{GL}}^{\Phi,\mathbf{M}}$ has the same axioms and rules as **GL** plus:
$$\Box\Box\Box\bot \tag{T3}$$
$$\Diamond\top \to \Diamond A \tag{C1}$$
$$\Diamond\Diamond\top \to \Diamond(B \wedge \Diamond C) \tag{C2}$$

In the axiom schemes C1 and C2, the formulas A, B and C all stand for consistent conjunctions of literals over Φ. [3] Note that $\mathbf{AX}_{\mathbf{GL}}^{\Phi,\mathbf{M}}$ is not a normal modal logic, because one cannot substitute just any formula for A, B, C. [4]

Definition 3.1 Define $M_{GL}^{\Phi} = (W, R, V)$, the *canonical asymptotic Kripke model* over Φ, with W, R, V as follows (see Fig. 1):
$W = \{b_v, m_v, u_v \mid v \text{ a propositional valuation on } \Phi\}$;
$R = \{\langle b_v, m_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\} \cup$
    $\{\langle m_v, u_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\} \cup$
    $\{\langle d_v, u_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\}$; and
for all $p_i \in \Phi$, the valuation V is defined by $V_{b_v}(p_i) = 1$ iff $v(p_i) = 1$; $V_{m_v}(p_i) = 1$ iff $v(p_i) = 1$; $V_{u_v}(p_i) = 1$ iff $v(p_i) = 1$.

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[3] C1 and C2 have been inspired by Carnap's consistency axiom: $\Diamond\varphi$ for any φ that is a consistent propositional formula [2], and used by Halpern and Kapron [8] for axiomatizing almost sure model validities for **K**-models.

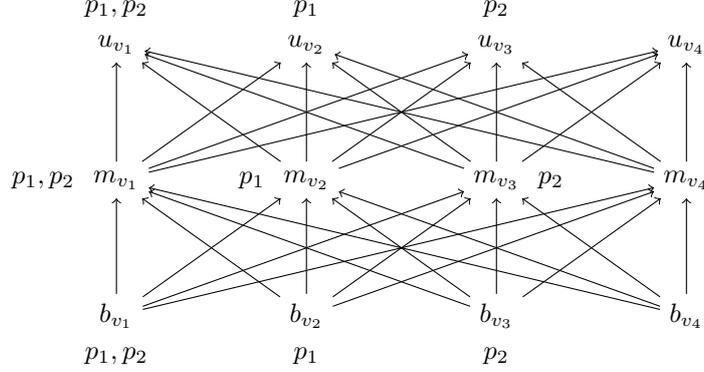[4] For example, substituting $\bot$ for A in C1 would make C1 equivalent to $\Box\bot$.

Fig. 1. The canonical asymptotic Kripke model $M_{GL}^{\Phi} = (W, R, V)$ for $\Phi = \{p_1, p_2\}$, defined in Definition 3.1. The accessibility relation is the transitive closure of the one drawn in the picture. The figure shows only propositional atoms true at each world.

The zero-one law for model validity now follows:

Theorem 3.2 *For every formula $\varphi \in L(\Phi)$, the following are equivalent:*

(i) $M_{GL}^{\Phi} \models \varphi$;

(ii) $\mathbf{AX_{GL}^{\Phi,M}} \vdash \varphi$;

(iii) $\lim_{n \to \infty} \nu_{n,\Phi}(\varphi) = 1$;

(iv) $\lim_{n \to \infty} \nu_{n,\Phi}(\varphi) \neq 0$.

3.2   Grz: 0-1 law in finite reflexive transitive anti-symmetric models

Define axiom system $\mathbf{AX_{Grz}^{\Phi,M}}$ as **Grz** plus the following axioms:

$$\neg(\varphi \wedge \Diamond(\neg\varphi \wedge \psi \wedge \Diamond(\neg\psi \wedge \chi \wedge \Diamond\neg\chi))) \tag{D3}$$

$$(\varphi \wedge \Diamond\neg\varphi) \to \Diamond A \tag{C3}$$

$$(\varphi \wedge \Diamond(\neg\varphi \wedge \psi \wedge \Diamond\neg\psi)) \to \Diamond(B \wedge \Diamond C) \tag{C4}$$

In the axiom schemes above, φ, ψ, χ stand for any formulas in $L(\Phi)$, while A, B and C stand for consistent conjunctions of literals over Φ. [5]

Definition 3.3 Define the canonical asymptotic Kripke model $M_{Grz}^{\Phi} = (W, R, V)$, where:
$W = \{b_v, m_v, u_v \mid v \text{ a propositional valuation on } \Phi\}$;
$R = \{\langle w, w \rangle \mid w \in W\} \cup$
     $\{\langle b_v, m_{v'} \rangle \mid v, v' \text{ propositional valuations on } \Phi\} \cup$
     $\{\langle m_v, u_{v'} \rangle \mid v, v' \text{ propositional valuations on } \Phi\} \cup$
     $\{\langle d_v, u_{v'} \rangle \mid v, v' \text{ propositional valuations on } \Phi\}$; and
$V_{b_v}(p) = 1$ iff $v(p) = 1$; $V_{m_v}(p) = 1$ iff $v(p) = 1$; $V_{u_v}(p) = 1$ iff $v(p) = 1$.

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[5] The axioms D3, C3 and C4 have been inspired by the axioms proposed in [8, Theorem 4.16] for the almost sure validities in finite **S4** models.

Note that M_{Grz}^{Φ} is just the reflexive closure of M_{GL}^{Φ} (Definition 3.1).

Theorem 3.4 *For every $\varphi \in L(\Phi)$, the following are equivalent: (i) $M_{Grz}^{\Phi} \models \varphi$; (ii) $\mathbf{AX}_{\mathbf{Grz}}^{\Phi,\mathbf{M}} \vdash \varphi$; (iii) $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 1$; (iv) $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) \neq 0$.*

3.3 wGRz: 0-1 law in finite transitive anti-symmetric models

Define the axiom system $\mathbf{AX}_{\mathbf{wGrz}}^{\Phi,\mathbf{M}}$ as **wGrz** plus axioms D3, C3 and C4.

Definition 3.5 The canonical asymptotic Kripke model M_{wGrz}^{Φ} is a combination of the irreflexive transitive M_{GL}^{Φ} and the reflexive transitive antisymmetric M_{Grz}^{Φ} (Def. 3.1 and 3.3), having a reflexive and irreflexive copy of each valuation-related world in each layer; it is transitive and antisymmetric and has direct accessibility from all states in the bottom layer to all states in the middle layer and all states in the middle layer to all states in the top layer.

Theorem 3.6 *For every $\varphi \in L(\Phi)$, the following are equivalent: (i) $M_{wGrz}^{\Phi} \models \varphi$; (ii) $\mathbf{AX}_{\mathbf{wGrz}}^{\Phi,\mathbf{M}} \vdash \varphi$; (iii) $\lim_{n\to\infty} \nu_n(\varphi) = 1$; (iv) $\lim_{n\to\infty} \nu_n(\varphi) \neq 0$.*

Conclusion

We have formulated zero-one laws for provability logic, Grzegorczyk logic and weak Grzegorczyk logic, with respect to model validity. On the way, we have axiomatized validity in almost all relevant finite models, leading to three axiom systems.[6] Many questions are left open for future research, most notably, those about almost sure frame validity.

References

[1] Boolos, G., "The Logic of Provability," Cambridge University Press, Cambridge, 1993.

[2] Carnap, R., "Meaning and Necessity," University of Chicago Press, Chicago, IL, 1947.

[3] Carnap, R., "Logical Foundations of Probability," University of Chicago Press, 1950.

[4] Compton, K. J., *The computational complexity of asymptotic problems I: Partial orders*, Information and Computation **78** (1988), pp. 108–123.

[5] Fagin, R., *Probabilities on finite models 1*, Journal of Symbolic Logic **41** (1976), pp. 50–58.

[6] Glebskii, Y. V., D. I. Kogan, M. Liogon'kii and V. Talanov, *Range and degree of realizability of formulas in the restricted predicate calculus*, Cybernetics and Systems Analysis **5** (1969), pp. 142–154.

[7] Grzegorczyk, A., *Some relational systems and the associated topological spaces*, Fundamenta Mathematicae **60** (1967), pp. 223–231.

[8] Halpern, J. Y. and B. Kapron, *Zero-one laws for modal logic*, Annals of Pure and Applied Logic **69** (1994), pp. 157–193.

[9] Kleitman, D. J. and B. L. Rothschild, *Asymptotic enumeration of partial orders on a finite set*, Transactions of the American Mathematical Society **205** (1975), pp. 205–220.

[10] Litak, T., *The non-reflexive counterpart of Grz*, Bulletin of the Section of Logic Univ. Lódz **36** (2007), pp. 195–208.

[11] Segerberg, K., "An Essay in Classical Modal Logic," Ph.D. thesis, Uppsala Univ. (1971).

[12] Verbrugge, R., *Zero-one laws for provability logic (abstract)*, in: *Proc. of the Tenth Int. Congress of Logic, Methodology and Philosophy of Science (LMPS95)*, 1995, pp. 79–80.

[13] Verbrugge, R., *Provability logic*, in: E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy (Fall 2017 Edition)*, Stanford University, 2017 .

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[6] Proofs can be found in the full paper.