NORTH-HOLLAND

A Qualitative Fuzzy Possibilistic Logic

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ABSTRACT

A formal logical system dealing with both uncertainty (possibility) and vagueness (fuzziness) is investigated. It is many-valued and modal. The system is related to a many-valued tense logic. A completeness theorem is exhibited.

KEYWORDS: ***fuzzy logic, modal logic, possibilistic logic, comparative approach***

1. INTRODUCTION

Many misunderstandings about fuzzy logic (in a narrow sense, i.e. fuzzy logic as a symbolic logical calculus) stem from the confusion between fuzziness as vagueness (impreciseness) on the one hand and uncertainty as partial belief (probabilistic or other) on the other hand, in spite of the fact that this difference has been stressed by several authors. Vagueness concerns degrees of truth and leads to many-valued logics, whereas uncertainty concerns degrees of belief and is best formalized as a sort of modal logic. See e.g. [9, 5, 11]. The main difference concerns the presence or absence of truth-functionality. Fuzzy logic deals with fuzzy propositions that may have intermediate degrees of truth (related often to values of some quantity such as height) and is usually understood as truth-

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functional, i.e., the truth degree of a compound formula (conjunction, disjunction, negation, etc.) is a function of the truth degrees of its components (computed using generalized truth tables). On the other hand, uncertainty as degree of belief about the truth of a crisp proposition is best understood as some measure (not necessarily probabilistic measure) on the set of all possible worlds (possible states, elementary events) assigning to a proposition p the measure of the set of all worlds in which p is true—and as such it is not truth-functional. We shall prefer possibility measures; and a possibility measure Π satisfies $\Pi(A \vee B) = \max(\Pi(A), \Pi(B))$, but the possibility of $A \wedge B$ is not a function of $\Pi(A)$, $\Pi(B)$. This is reminiscent of modal logic with modalities $\Diamond$ (possibly) and $\Box$ (necessarily). For most modal systems, $\Diamond(A \vee B)$ is equivalent to $\Diamond A \vee \Diamond B$, but $\Diamond(A \wedge B)$ is not equivalent to $\Diamond A \wedge \Diamond B$. The mathematical framework to define semantics of modal logic—Kripke models—generalizes often to systems with other modalities.

Even if formulas bear possibilities, which are numerical values, we may be interested not in the values themselves but in their comparison, i.e. investigate formulas $A \lhd B$ saying that B is at least as possible as A. Here $\lhd$ behaves as a modality generalizing in some sense the modality $\Diamond$ (possibly). The logical language using $\lhd$ is qualitative (or, we can say, comparative), i.e., it does not have means to express possibilities as numbers, but only their comparisons. This leads to interesting, well-defined, and natural logical systems, and the question naturally emerges if they are related to some classical systems of modal logic. Developing formal logics like ours, we show what certain and crisp statements can be made about uncertainty and fuzziness. A completeness result shows that our axiomatization completely grasps truth (1-tautologicity). And a natural embedding of a system of a logic of uncertainty into a more "classical" modal logic (both fuzzy) shows that our system fits well into the population of established modal logic systems, which supports our belief that the system is sound (well defined, adequately formalizing some aspects of uncertainty and vagueness).

In [1] a qualitative possibilistic modal logic is studied and related to a tense (temporal) logic with finite linearly preordered time. [1] is related to [6]; among other things, the system QPL of [6] is shown to be incomplete (but complete for formulas without nested modalities), and a complete system is presented. A complementary paper is [3]; this paper also relates modal possibilistic logic to a (different) tense logic and discusses [6], but e.g. the above-mentioned results of [1] on incompleteness and completion of QPL do not occur in [3] (and various results of [3] do not occur in [1]). Here we analyze a logical system dealing with both fuzziness and uncertainty, which means, formally, that it is both many-valued and modal. Many-valued modal logics have been investigated by some authors (see [16, 7, 8]); this makes our investigation easier. Trying to build a system of qualitative fuzzy possibilistic logic, we immediately meet the problem how to compare formulas of fuzzy logic with respect to their possibility or, more generally, with respect to possibilities of worlds on which they have a given truth value. This can be done in various ways; we develop two of them and show that they are closely related to each other and also to a system of many-valued tense logic (with reflexive linearly preordered time).

The reader may wonder why we speak of tense logics in relation to possibilistic logic. The primary reasons is, admittedly, formal: structures defining the semantics of possibilistic logics lead naturally to structures defining the semantics of various tense (temporal) logics, i.e. to structures where the set of possible worlds bears a (pre)order. Relating our logical systems formally to some well-established systems is a desirable thing, witnessing mathematical soundness of our systems. But in our opinion one may learn even more: relating possibilistic logic to tense logic may improve our intuitive understanding of what possibilities (of the possibility theory) are: one interpretation of $\Pi(A)$ is the last time moment at which A is true (in the infinite case, the supremum of these moments).

We mention the paper [13], where a definition of the possibility of a fuzzy formula is given that is drastically different from that used in the present paper; this leads to a fuzzy modal logic unrelated (apparently) to any tense logic, but related to fuzzy variants of S5 (logic of knowledge) and KD45 (logic of belief). The rest of the paper is organized as follows: In Section 2 we elaborate the definition of our two calculi of qualitative fuzzy logic and prove some theorems about them; in Section 3 we introduce our many-valued tense logic, state a completeness theorem, and show a relation to our fuzzy logic. Section 4 contains the main theorem, showing that our two systems, in spite of different semantics, have the same tautologies and extend conservatively to our tense logic. Section 5 summarizes our conclusions and presents some additional information (including a proof of the completeness theorem).

A preliminary short version of this paper appeared as [12]. Note that the tense logic in the present paper differs from that of [12]; see Remark 4.2 for a comparison.

2. TWO QUALITATIVE FUZZY LOGICS

2.1. Values, Symbols, Formulas

We fix $n + 1 \geq 2$—the number of truth values. Values $= \{0, 1/n, 2/n, \ldots, 1\}$ is the set of values. Our *symbols* are propositional variables, connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$ (conjunction, disjunction, implication, negation), and a unary connective (i) for each $i \in$ Values; modalities will be introduced later. Modality-free formulas are built from propositional variables using connectives. For example, $A \to (1)A$ is a formula.

2.2. Truth Tables

We shall use Łukasiewicz's truth tables for connectives: minimum and maximum for conjunction and disjunction respectively, implication $I(i, j) = \min(1, 1 - i + j)$, negation $N(i) = 1 - i$; for each i, the truth table δ_i of the connective (i) assigns 1 to i and 0 to each $j \neq i$. (Note that these connectives are in fact definable in Łukasiewicz's logic; cf. [10, 16]. However, for technical reasons we taken them as primitive.) Most of our investigations are valid also for other choices of semantics of connectives, e.g. Gödel's logics [10]; but we shall not rely on this.

2.3. Kripke Structures

A fuzzy possibilistic Kripke structure is a structure $K = \langle W, \Vdash, \pi \rangle$ where W is a nonempty set of worlds, $\Vdash$ maps Atoms $\times$ W into Values (thus each atom has a value in each world), and π is a normalized positive fuzzy subset of W, i.e. a mapping $\pi : W \to [0, 1]$ such that $\pi(w) > 0$ for each w and $\sup_{w \in W} \pi(w) = 1$ (thus "impossible possible worlds" are not allowed). We call $\pi(w)$ the *possibility* of the world w; the possibility of a set $X \subseteq W$ of worlds is defined as $\Pi(X) = \sup(\pi[X]) = \sup_{w \in X} \pi(w)$. We write $\|p\|_w$ instead of $\Vdash(p, w)$ and extend $\Vdash$ to all Boolean combinations of atoms using truth tables; thus e.g. $\|A \wedge B\|_w = \min(\|A\|_w, \|B\|_w)$, $\|(i)A\|_w = 1$ iff $\|A\|_w = i$, $\|(i)A\|_w = 0$ otherwise, etc.

Besides fuzzy possibilistic Kripke structures we shall investigate also *fuzzy tense Kripke structures* $\langle W, \Vdash, \leq \rangle$ where W and $\Vdash$ are as above and $\leq$ is a reflexive linear preorder of W, i.e., $\leq$ is reflexive, transitive, and connected: $(\forall w, v \in W)(w \leq v$ or $v \leq w)$. Clearly, each fuzzy possibilistic structure $\langle W, \Vdash, \pi \rangle$ determines a tense structure $\langle W, \Vdash, \leq_\pi \rangle$ where

$$w \leq_\pi v \quad \text{iff} \quad \pi(w) \leq \pi(v) \quad \text{(as reals)}.$$

2.4. Sets of Worlds

Given W and $\Vdash$ as above and a truth value i, each (modality-free) formula determines the set worlds(A, i) of all worlds w such that $\|A\|_w = i$ [or, equivalently, $\|(i)A\|_w = 1$]. Similarly, worlds$(A, \geq i)$ is the set of all w such that $\|A\|_w \geq i$. Clearly, if we want to compare formulas according to their possibilities, these sets of worlds will be relevant. Thus let us concentrate, for a moment, on comparison of sets of worlds.

2.5. Comparing Sets of World

If $\langle W, \pi \rangle$ is a set of worlds with a normalized positive fuzzy subset (say a possibilistic set), then subsets X, Y of W may be compared according to their possibilities: write $X \preccurlyeq Y$ if $\Pi(X) \leq \Pi(Y)$. If $\langle W, \leq \rangle$ is a reflexive linearly preordered set, then subsets $X, Y \subseteq W$ may be compared according to dominance: X is *dominated* by Y (write $X \preccurlyeq' Y$) if $(\forall x \in X)(\exists y \in Y)(x \leq y)$. If we speak of dominance in a possibilistic set $\langle W, \pi \rangle$, we mean dominance on the corresponding reflexive linearly preordered set tense$(W, \pi) = \langle W, \leq_\pi \rangle$. We state some easy properties of $\preccurlyeq$, $\preccurlyeq'$.

LEMMA 2.1
(1) *Both* $\preccurlyeq$ *and* $\preccurlyeq'$ *are reflexive linear preorders* (*i.e. are reflexive, transitive and connected*).
(2) *For any* $\langle W, \pi \rangle$, $X \preccurlyeq' Y$ *implies* $X \preccurlyeq Y$; $X \subseteq Y$ *implies* $X \preccurlyeq' Y$.
(3) *For a finite* $\langle W, \pi \rangle$ (*i.e.*, W *is finite*), $\preccurlyeq$ *and* $\preccurlyeq'$ *coincide.*
(4) *For some infinite* $\langle W, \pi \rangle$ $\preccurlyeq$ *and* $\preccurlyeq'$ *do not coincide.*

Proof The easy proofs are left to the reader (to prove (3), observe that if X and Y are finite then $X \preccurlyeq' Y$ iff $\max X \leq \max Y$; to prove (4) produce a W, π, X, Y such that $\sup(\pi[X]) = \sup(\pi[Y]) = \alpha$, $\alpha \in \pi[X]$, $\alpha \notin \pi[Y]$; e.g. $W = X = [\frac{1}{2}, 1]$, $\pi(w) = w$, $Y = X - \{1\}$). ■

But for some infinite $\langle W, \pi \rangle$ the two preorders do coincide; here is an example.

LEMMA 2.2 *A set* $Z \subseteq [0, 1]$ *is said to be* isolated from below *if for each* $z \in Z$ *there is* $\varepsilon > 0$ *such that* $[z - \varepsilon, z] \cap Z = \{z\}$, *i.e., the only* $t \in Z$ *such that* $z - \varepsilon \leq t \leq z$ *is* z *itself. If* $\langle W, \pi \rangle$ *is such that* $\pi[W] = \{\pi(w) \mid w \in W\}$ *is isolated from below, then the corresponding preorders* $\preccurlyeq$ *and* $\preccurlyeq'$ *coincide.*

(This is because for Z isolated from below, if $y \in Z$, $X \subseteq Z$, and all elements of X are less than y, then $\sup X < y$.)

LEMMA 2.3 *There is a countable set* $Z \subseteq [0, 1]$ *which is isolated from below and is densely ordered by the usual ordering of reals, i.e.,* $(\forall z_1, z_2 \in Z)\,(z_1 < z_2 \rightarrow (\exists z_0)\,(z_1 < z_0 < z_2))$.

Proof Hint: Take all positive rational numbers in the unit interval whose decimal expansion is finite and contains only digits 0 and 7. ■

COROLLARY 2.1 *If* $\langle W, \leq \rangle$ *is a countable reflexive linearly preordered set, then there is a positive normalized* π *on* W *such that* $\leq$ *is the preorder* $\leq_\pi$ *defined by* π *and* $\pi[W]$ *is isolated from below, i.e., for* $\langle W, \pi \rangle$ *the preorders* $\preccurlyeq$ *and* $\preccurlyeq'$ *coincide.*

2.6. Comparing Formulas

Recall the sets worlds(A, i) and worlds(A, $\geq i$) given by a formula A and a truth value i. We shall introduce two binary modalities $\lhd$ and $\lhd'$ such that, for each $K = \langle W, \Vdash, \pi \rangle$

$$\|A \lhd B\|_w = 1 \quad \text{iff} \quad (\forall i)(\text{worlds}(A, \geq i) \preccurlyeq \text{worlds}(B, \geq i)),$$

$$\|A \lhd' B\|_w = 1 \quad \text{iff} \quad \text{the same, with } \preccurlyeq' \text{ instead of } \preccurlyeq .$$

It is clear that $\|A \lhd B\|_w$ and $\|A \lhd' B\|_w$ are independent of w. In the rest of the paper, when $\|A\|_w$ does not depend on w, we will often simply write $\|A\|$.

If $\|A \lhd B\| = 1$, we say that A is *at most as possible as B*; if $\|A \lhd' B\| = 1$, we say that A is *dominated by B*.

Remark 2.1

(1) $\|A \lhd B\| = 1$ means that for each i, the possibility of A having value *at least i* (i.e. being sufficiently true, say) is at most as big as the corresponding possibility for B. This is more natural than a similar condition but for value *exactly i* instead of at least i [thus using worlds(A, i)]. Moreover, as the reader easily checks, $\|A \to B\| = 1$ implies $\|A \lhd B\| = 1$ and $\|A \lhd' B\| = 1$ for the present definition; this would fail after the modification indicated. So we keep our definition.

(2) The definition of $\lhd$ may be expressed as comparison of *fuzzy truth values*: put, for each $i \in$ Values,

$$\sigma_A(i) = \Pi(\text{worlds}(A, \geq i)).$$

Then $\|A \lhd B\| = 1$ iff $(\forall i)(\sigma_A(i) \leq \sigma_B(i))$, and σ_A, σ_B are fuzzy truth values (fuzzy subsets of the set of values).

(3) As one of the referees remarked, the function σ_A might be used to define other modalities, e.g. by $(\exists i > 0)(\sigma_A(i) \leq \sigma_B(i))$. This could be a topic of further research.

EXAMPLE 2.1 The table below represents a Kripke model with seven worlds ($W = \{1, 2, 3, 4, 5, 6, 7\}$), two propositions p, q, and possibility distribution π. Since p is true in world 7, whose possibility is 1, we have $\sigma_p(i) = 1$ for all $i = 0, \frac{1}{2}, 1$; but $\sigma_q(1) = 0.6$, since the world of maximal possibility in which q is true is world 4 and its possibility is 0.6. The reader may check that $\|q \lhd p\| = 1$ but $\|p \lhd q\| \neq 1$ (see also below):

	p	q	π
1	1	1	0.3
2	$\frac{1}{2}$	1	0.4
3	0	$\frac{1}{2}$	0.5
4	1	1	0.6
5	1	$\frac{1}{2}$	0.7
6	1	0	0.8
7	1	$\frac{1}{2}$	1.0
	0	$\frac{1}{2}$	1
σ_p	1	1	1
σ_q	1	1	0.6

2.7. Fuzzifying: Fuzzy Modalities Defined

Till now, we have only defined what it means for $A \lhd B$ and $A \lhd' B$ to have value 1. Now we complete our definition of semantics of these modalities. This may be done in various ways; we could e.g. make the formulas $A \lhd B$, $A \lhd' B$ Boolean and declare them to have value 0 if they do not have value 1. But we prefer something more fuzzy (for reasons that become apparent later). Observe the following:

LEMMA 2.4 $\|A \lhd B\| = 1$ *iff*

$$(\forall i)(\exists j \geq i)(\text{worlds}(A, i) \preccurlyeq \text{worlds}(B, j)), \qquad (*)$$

and similarly for $\lhd'$, $\preccurlyeq'$.

Proof It follows from the finiteness of Values that for each $i \in$ Values there is a $j \geq i$ such that the set worlds(B, $\geq i$) is equivalent, with respect to $\preccurlyeq$, to worlds(B, j) (more generally, for any X, Y, $X \cup Y$ is $\preccurlyeq$ -equivalent to X or to Y). If worlds(A, $\geq i$) $\preccurlyeq$ worlds(B, $\geq i$) then worlds(A, i) $\preccurlyeq$ worlds(A, $\geq i$) $\preccurlyeq$ worlds(B, $\geq i$) $\preccurlyeq$ worlds(B, j) for some $j \geq i$. Conversely, if the condition $(*)$ holds, then worlds(A, $\geq i$) $\preccurlyeq$ worlds(A, i') for some $i' \geq i$; hence for some $j \geq i' \geq i$, worlds(A, $\geq i$) $\preccurlyeq$ worlds(A, i') $\preccurlyeq$ worlds(B, j) $\preccurlyeq$ worlds(B, $\geq i$). ■

LEMMA 2.5 *Note that* $j \geq i$ *iff* $I(i, j) = 1$ (*the truth table of implication*); *thus* $\|A \lhd B\| = 1$ *iff*

$$\min_i \max_j \{I(i, j) \mid \text{worlds}(A, i) \preccurlyeq \text{worlds}(B, j)\} = 1.$$

For given i, $\max_j\{I(i, j) \mid \text{worlds}(A, i) \preccurlyeq \text{worlds}(B, j)\}$ *measures how big, with respect to* i, *is the maximal* j *such that* $\Pi(\text{worlds}(B, j)) \geq \Pi(\text{worlds}(A, i))$.

Note that there is always at least one j such that $\Pi(\text{worlds}(B, j)) \geq \Pi(\text{worlds}(A, i))$, since for some j we have $\Pi(\text{worlds}(B, j)) = 1$. The bigger is the maximal j satisfying the above, the bigger is the value of the implication $I(i, j)$; this value is 1 iff $i \leq j$. Thus $\max_j\{I(i, j) \mid \text{worlds}(A, i) \preccurlyeq \text{worlds}(B, j)\}$ may be understood as the truth value of the fuzzy statement "for some truth degree j not much less than i, $\Pi(B, j) \geq \Pi(A, j)$." The quantity $\min_i \max_j\{I(i, j) \mid \text{worlds}(A, i) \preccurlyeq \text{worlds}(B, j)\}$ then can be understood as the truth value of the statement "for each truth value i, there is a j not much less then i such that $\Pi(B, j) \geq \Pi(A, j)$." Yet fuzzier: "the truth of B is almost as possible as the truth of A."

Similarly for dominance.

This leads us to the following

DEFINITION 2.1

$$\|A \lhd B\| = \min_i \max_j \{I(i, j) \mid \text{worlds}(A, i) \preccurlyeq \text{worlds}(B, j)\},$$

$$\|A \lhd' B\| = \text{the same, with } \preccurlyeq' \text{ instead of } \preccurlyeq .$$

We can read $A \lhd B$ as "B is almost at least as possible as A," and $A \lhd' B$ as "B almost dominates A." If the value is 1, the word "almost" becomes superfluous.

EXAMPLE 2.2 In Example 2.1 verify that $\|p \lhd q\| = \frac{1}{2}$.

2.8. The Calculi Defined

Now we are finally ready to complete our definition of two qualitative fuzzy logics:

DEFINITION 2.2

(1) *The* qualitative fuzzy possibilistic logic QFL *has connectives and truth tables as above and has the modality* $\lhd$ *(comparing of possibilities); its models are fuzzy possibilistic Kripke structures* $\langle W, \Vdash, \pi \rangle$. *To repeat once more,* $\|A \lhd B\| = 1$ *if for each* i, *the possibility of* A *having the value* $\geq i$ *is less than or equal to the possibility of* B *having value* $\geq i$.

(2) *The* qualitative fuzzy dominance logic QFL′ *has connectives and truth tables as above and has the modality* $\lhd'$ *(dominance); its models are fuzzy tense Kripke structures* $\langle W, \Vdash, \leq \rangle$. *(Note that among them are models* tense$(\langle W, \Vdash, \pi \rangle)$ *given by possibilistic*

models.) To repeat: $\|A \lhd' B\| = 1$ *if for each i, worlds in which A has value* $\geq i$ *are dominated by worlds in which B has value* $\geq i$.

Remark 2.2 Let us note explicitly that we allow nested occurrences of modalities, e.g., $p \lhd (p \lhd p)$ is a formula.

LEMMA 2.6 *For each fuzzy possibilistic Kripke model* $K = \langle W, \Vdash, \pi \rangle$ *and each formula A of* QFL, *there is a finite model* $K' = \langle W', \Vdash', \pi' \rangle$ *and a surjection f of W onto W' such that, for each* $w \in W$,

$$\|A\|_{K,w} = \|A\|_{K',f(w)}.$$

Proof Let $p_1, \dots, p_n$ be all the propositional variables occurring in A; an *elementary conjunction* (EC) is a formula of the form $(i_1)p_1 \wedge \cdots \wedge (i_n)p_n$ where $i_1, \dots, i_n \in$ Values. Observe that in each world w exactly one EC has the value 1 and others have the value 0. For each $w \in W$ let $f(w)$ be that EC true in w, and let $W' = f[W] = \{f(w) \mid w \in W\}$. Clearly, W' is finite (since Values is a finite set) and nonempty. For each p_i, define its value in a $w' \in W'$ to be j if the formula $(j)p_i$ is a subformula of w' (recall that w' is an EC), and assign an arbitrary value to propositional variables other than $p_1, \dots, p_n$. For each $w' \in W'$, let $\pi'(w') = \Pi_K(\text{worlds}(w', 1))$, i.e. the possibility of the elementary conjunction w' having value 1 in K. This defines $K' = \langle W', \Vdash', \pi' \rangle$. Now one can verify by induction on the complexity of the formula that for each subformula B of A,

$$\|B\|_{K,w} = \|B\|_{K',f(w)}, \tag{1}$$

$$\Pi_K(\text{worlds}(B, i)) = \Pi_{K'}(\text{worlds}(B, i)). \tag{2}$$

In particular, if (1), (2) hold for B and C (and all i), then clearly $\|B \lhd C\|_K = \|B \lhd C\|_{K'}$ (independently of w); this is the induction step for $\lhd$. ■

Remark 2.3 The analogous lemma holds for *countable* tense models and QFL′ formulas thanks to Corollary 2.1 (and the fact that each countable linear order is isomorphically embeddable into the countable dense order without endpoints). We shall prove a weaker finiteness theorem for QFL′ later (see Theorem 4.1).

3. A MANY-VALUED TENSE LOGIC

We are going to relate our qualitative fuzzy logics to a many-valued tense logic with reflexive linearly preordered time. The logic in question, denoted by MTL, will have the same modality-free formulas, truth values, and truth tables as the logic QFL described in the previous section; but it will have two unary modalities G, H (read "in all future and present worlds," "in all past and present worlds" respectively).

Kripke models have the form $\langle W, \Vdash, \leq \rangle$, where $\leq$ is a reflexive linear preorder on W.

The semantics of the modality G is as follows:

$$\|GA\|_w = \inf_{w' \geq w} \|A\|_{w'},$$

and analogously for H ($\inf_{w' \leq w}$). We further define $FA \equiv \neg G(\neg A)$, $PA \equiv \neg H(\neg A)$, $\Box A \equiv GA \wedge HA$, $\Diamond A \equiv \neg \Box(\neg A)$. The formulas FA, PA, $\Box A$, $\Diamond A$ are read: A holds in some future or present world, some past or present world, in all worlds, in some worlds.

Caution Observe that the semantics of the modalities is defined differently from [12], where we worked with *strict* linear preorder $<$. Thus, again, our modalities are "always from now on" and "always until now," and hence are hidden universal quantifications over possible worlds $\geq w$ or $\leq w$. Hence the interpretation by infimum is very standard, and we just generalize usual two-valued tense logics with linearly ordered time (working with finitely many truth values and with linear preorders instead linear orders; thus in one time moment there may be several alternative possible worlds).

DEFINITION 3.1 *A formula is called* Boolean *if it results from formulas of the form* $(i)A$ *using connectives and modalities. Clearly, if* B *is Boolean then* $\|B\|_w$ *is* 0 *or* 1.

AXIOMS

(a) *Propositional axioms.* Some choice of axioms complete for the given propositional calculus (cf. [10]): some few axioms for implication and other connectives; furthermore,

(1) $\bigvee_i (i)A$,

(2) $\bigwedge_{i \neq j} \neg((i)A \wedge (j)A)$ (saying that each formula has exactly one truth value),

(3) $(i)A \wedge (j)B \rightarrow (\min(i, j))(A \wedge B)$, and similarly for other connectives $\vee$, $\rightarrow$, $\neg$, e.g.

(3′) $(i)A \wedge (j)B \rightarrow (1)(A \rightarrow B)$ if $i \leq j$, etc.;

(3″) $(1)A \rightarrow A$ for arbitrary A, and also $A \equiv (1)A$ for A Boolean;

(3‴) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ for A, B, C Boolean.

Note that (3‴) is not sound for all formulas of Łukasiewicz's logic, but for Boolean ones it is. (3‴) is one of usual axioms of the classical propositional calculus, and it guarantees that for Boolean formulas we have full classical propositional calculus.

(b) *Modal axioms.*

(4) $G(A \to B) \to (GA \to GB)$, $H(A \to B) \to (HA \to HB)$; $PGA \to A$, $FHA \to A$;

(5) $GA \to A$ (reflexivity); $GA \to GGA$ (transitivity);

(6) $FA \to G(PA \vee FA)$ (not branching towards the future); $PA \to H(PA \vee FA)$ (not branching towards the past);

(7) $G(\geq i)A \equiv (\geq i)GA$, $G(\leq i)A \equiv (\leq i)FA$, $H(\geq i)A \equiv (\geq i)HA$, $H(\leq i)A \equiv (\leq i)PA$ (an analog of Fitting's axioms [7, 8]).

[Note that $(\geq i)A$ is $\bigvee_{j \geq i}(j)A$ etc.]

DEDUCTION RULES of this modal logic *Modus ponens, necessitation for* G, H *(e.g., "from* A *infer* GA*"), and also "from* A *infer* $(1)A$*."*

Caution To avoid misunderstanding, let us stress again that some axioms are assumed only for Boolean formulas, i.e. formulas whose syntactic form guarantees that they take only values 0, 1 in all evaluations. Only for such formulas do we have full propositional calculus. But all axioms and deduction rules are 1-sound, i.e., axioms are 1-tautologies (true in all worlds of all Kripke models with reflexive linear ordering), and all deduction rules convert 1-tautologies to 1-tautologies. This is the content of the following lemma.

LEMMA 3.1 *All the above axioms and deduction rules are* 1*-sound for Kripke models with reflexive linearly preordered time.*

Proof (3″): $\|(1)A\|_w = 1$ iff $\|A\|_w = 1$, $\|(1)A\|_w = 0$ otherwise; thus $\|(1)A\|_w \leq \|A\|_w$ for arbitrary A. If A is Boolean, i.e. has only values 0, 1, then $\|(1)A\|_w = \|A\|_w$.

(4): Consider $G(A \to B) \to (GA \to GB)$. Nothing has to be proved if $\|GA\|_w \leq \|GB\|_w$, so assume $\|GA\|_w > \|GB\|_w$. Let w_1, w_2 be such that $\|GA\|_w = \|A\|_{w_1} = a_1$, $\|GB\|_w = \|B\|_{w_2} = b_2$; put $\|A\|_{w_2} = a_2$. Then $a_2 \geq a_1 > b_2$, so $\|GA \to GB\|_w = 1 - a_1 + b_2 \geq 1 - a_2 + b_2 \geq \|G(A \to B)\|_w$.

Consider $PGA \to A$; let $a = \|A\|_w$. For each $w' \leq w$, $\inf_{w'' \geq w'} \|A\|_{w''} \leq a$; thus $\|PGA\|_w \leq a$.

The proofs for the other two axioms are similar.

(5): Consider $GA \to A$. By definition, $\|GGA\|_w = \inf_{w' \geq w} \|A\|_{w'}$, so because $w \geq w$, $\|GA\|_w \leq \|A\|_w$.

Consider $GA \to GGA$. By definition, $\|GGA\|_w = \inf_{w' \geq w} \inf_{w'' \geq w'} \|A\|_{w''}$. By transitivity, if $w'' \geq w'$ and $w' \geq w$ then $w'' \geq w$, so $\|GA\|_w \leq \|GGA\|_w$.

(6): Consider $FA \to G(PA \vee FA)$. Let $a = \|FA\|_w$, and suppose that $a > 0$ (otherwise nothing has to be proved). Let w_1 be such that $\|FA\|_w =$

$\|A\|_{w_1}$, and let $w_2 \geq w$ be arbitrary. Because $\leq$ is not branching towards the future, there are two possibilities: $w_1 \leq w_2$ and thus $\|PA\|_{w_2} \geq a$, or $w_2 \leq w_1$ and thus $\|FA\|_{w_2} \geq a$. In both cases, $\|PA \vee FA\|_{w_2} \geq a$. Therefore $\|G(PA \vee FA)\|_w \geq a$, as desired.

The proof for the other axiom is similar.

(7): Consider $G(\leq i)A \equiv (\leq i)FA$. We have $\|G(\leq i)A\|_w = 1$ iff $\min_{w' \geq w} \|(\leq i)A\|_{w'} = 1$ iff for all $w' \geq w$, $\|A\|_{w'} \leq i$, iff $\max_{w' \geq w} \|A\|_{w'} \leq i$, iff $\|FA\|_w \leq i$, iff $\|(\leq i)FA\|_w = 1$.

Finally we prove 1-soundness of necessitation by G: if A is a 1-tautology, then A has value 1 in each world w of each Kripke model K; then also GA has value 1 in w (since it has value 1 in all $w' \geq w$ from K).

The rest is similar. ■

Remark 3.1 Axioms (4)–(6) are very similar to the tense logic Lin defined in [14]. Because we added the reflexivity axiom, the axioms (6) are somewhat simpler than the ones usually used ($FA \rightarrow G(PA \vee A \vee FA)$ and $PA \rightarrow H(PA \vee A \vee FA)$; see also [2]). The axioms (7) are inspired by Fitting [7], even if not identical with his axioms.

THEOREM 3.1 (Completeness theorem) MTL $\vdash A$ *(A is provable in our tense logic) iff A is a 1-tautology of* MTL, *i.e., A has the value* 1 *in all worlds of all Kripke models with reflexive linearly preordered time.*

The proof is a variant of the standard method of canonical models for two-valued tense logic (cf. [14]), and is presented in Section 5. For general information on tense logics and corresponding proof techniques see [2, 4].

Our proof, combined with the fact that MTL satisfies the finite-model property (see Lemma 5.9 below), in fact gives the following

LEMMA 3.2 *A formula A is not provable in* MTL *if and only if there is a Kripke structure $\langle W, \Vdash, \leq \rangle$ with W finite and such that for some $w \in W$, $(< 1)A$ is true in w.*

The following lemma relates QFL$'$ to MTL.

LEMMA 3.3 *For an arbitrary tense model $K = \langle W, \Vdash, \leq \rangle$ with reflexive linearly preordered time and any pair A, B of formulas,*

$$\|A \lhd' B\|_K = \|\Box(A \rightarrow FB)\|_K.$$

Proof For each $w \in W$, put $\beta(w) = \max_{w' \geq w} \|B\|_{w'}$; β is nonincreasing. For each $i \in$ Values, let

$$j_i = \min_{\|A\|_w = i} \beta(w) = \min_{\|A\|_w = i} \max_{w' \geq w} \|B\|_{w'},$$

Observe that for each w,

$$\|A\|_w = i \to (\exists w' \geq w)(\|B\|_{w'} = j_i)$$

[if $\beta(w) = j_i$ then $j_i = \max_{w' \geq w} \|B\|_{w'}$; if $\beta(w) > j_i$, then there is a $w_0 > w$ (i.e., $w_0 \geq w$ but not $w \geq w_0$) such that $\beta(w_0) = j_i$]. We get

$$\text{worlds}(A, i) \preccurlyeq' \text{worlds}(B, j_i);$$

and if $j > j_i$, then there is a w with $\|A\|_w = i$ and $(\forall w' \geq w)(\|B\|_{w'} < j)$. Hence for each i, j_i is the maximal j satisfying worlds$(A, i) \preccurlyeq'$ worlds(B, j).

Now $\|\Box(A \to FB)\|_K = \min_w I(\|A\|_w, \|FB\|_w) = \min_w I(\|A\|_w, \beta(w)) = \min_i[\min_{\|A\|_w = i} I(i, \beta(w))] = \min_i[I(i, j_i)] = \min_i \max_j\{I(i, j) \mid \text{worlds}(A, i) \preccurlyeq' \text{worlds}(B, j)\} = \|A \lhd' B\|_K$. ■

4. MAIN THEOREM

In this section we show how closely related are our two fuzzy logics QFL and QFL′ to each other and to the tense logic MTL. For each QFL formula Φ we define its QFL′ variant Φ' and the translation Φ^* of Φ into the language of MTL. Then we show that Φ has a model iff Φ' has iff Φ^* has. Let us make the necessary definitions.

DEFINITION 4.1 *For each* QFL *formula* Φ, *let* Φ' *be a* QFL′ *formula which results from* Φ *on replacing each occurrence of* $\lhd$ *by* $\lhd'$; *and let* Φ^* *be an* MTL *formula which results from* Φ *on successively replacing each subformula* $A \lhd B$ *by* $\Box(A \to FB)$ (*in more detail,* Φ^* *is* Φ *for* Φ *atomic,* * *commutes with connectives, and* $(A \lhd B)^*$ *is* $\Box(A^* \to FB^*)$).

Remark 4.1 By "Φ has a QFL model" we mean the following: there is a Kripke structure $K = \langle W, \Vdash, \pi \rangle$ and a world $w \in W$ such that $\|\Phi\|_{K,w} = 1$. Note that 1 can be replaced with any i by replacing Φ with $(i)\Phi$; we have $\|\Phi\|_{K,w} = i$ iff $\|(i)\Phi\|_{K,w} = 1$. Similarly for $(< i)\Phi$ etc.

THEOREM 4.1 (Main theorem) *For each* QFL *formula* Φ, *the following are equivalent*:

(1) Φ *has a* QFL *model.*
(2) Φ' *has a* QFL′ *model.*
(3) Φ^* *has a* MTL *model.*
(4) Φ *has a finite* QFL *model.*
(5) Φ' *has a finite* QFL′ *model.*
(6) Φ^* *has a finite* MTL *model.*

Proof (1) $\Leftrightarrow$ (4) by the finiteness Lemma 2.6; (4) $\Leftrightarrow$ (5) by Lemma 2.1(3); (5) $\Leftrightarrow$ (6) by Lemma 3.3; (6) $\Leftrightarrow$ (3) by Lemma 3.2; (3) $\Leftrightarrow$ (2) by Lemma 3.3. This completes the proof. ■

COROLLARY 4.1 *The following are equivalent*:

(1) Φ *is a* 1-*tautology of* QFL.
(2) Φ' *is a* 1-*tautology of* QFL$'$.
(3) Φ^* *is a* 1-*tautology of* MTL.
(4) Φ *is* 1-*true in each world of each finite* QFL-*model.*
(5) Φ' *is* 1-*true in each world of each finite* QFL$'$-*model.*
(6) Φ^* *is* 1-*true in each world of each finite* MTL-*model.*

Remark 4.2 Thus both QFL and QFL$'$ have the finite-model property: a formula has a model iff it has a finite model. (Note that the mapping associating with each QFL formula the corresponding QFL$'$ formula maps QFL formulas *onto* QFL$'$ formulas.)

We see that the mapping * makes MTL a *conservative extension* of QFL (and similarly for QFL$'$); Φ is a 1-tautology of QFL iff Φ^* is a 1-tautology of MTL.

Let us note at this point that in [12] a different tense logic MTL* is used, with three basic modalities "in all future worlds," "in all present worlds," "in all past worlds." A similar (two-valued) logic was used in [1]; [3] has a two-valued logic with two modalities, one nonstrict (we would say "in all present or future worlds") and one strict ("in all past worlds"; but Boutilier does not speak of tense logics). Clearly, MTL* is stronger (more expressive) than MTL; in particular, MTL* does *not* have the finite-model property. (Neither the logic of Bendová and Hájek nor that of Boutilier has the finite-model property.) Nevertheless, for (interpretations of) QFL formulas, both MTL and MTL* have the same strength: both extend QFL (as well as QFL$'$) conservatively.

5. APPENDIX

We prove the completeness theorem for MTL and close with some remarks.

LEMMA 5.1 *The following formulas are provable in* MTL:

(11) $(\geq i)A \to ((1)(A \to B) \to (\geq i)B)$;
(12) $GB \to (FA \to F(A \wedge B))$;
(13) $F(\geq i)A \equiv (\geq i)FA$ *for* $i > 0$; $F(\geq 0)A \equiv F(\text{true})$;
(14) $F(\leq i)A \equiv (\leq i)GA$ *for* $i < 1$; $F(\leq 1)A \equiv F(\text{true})$;
(15) $(i)GA \equiv G(\geq i)A \wedge F(i)A$ *for* $i < 1$; $(1)GA \equiv G(1)A$;
(16) $(i)FA \equiv G(\leq i)A \wedge F(i)A$ *for* $i > 0$; $(0)FA \equiv F(0)A$.

Similarly for the other modalities.

Proof Hints: (13) and (14) follow from (7); (15) follows also from (7) using $\vdash (i)GA \equiv (\geq i)GA \wedge (\leq i)GA$ and $\vdash G(\geq i)A \rightarrow (F(\leq i)A \rightarrow F(i)A)$ [by (12)]. The proof of (16) is similar. ■

DEFINITION 5.1 *A* theory *is a set of Boolean formulas containing all* MTL*-provable Boolean formulas (in particular, T contains $(1)A$ for each* MTL*-provable formula A). A Boolean formula A is provable in T (notation $T \vdash A$) if it has a propositional proof from members of T using only modus ponens (i.e., necessitation is not allowed). T is inconsistent if $T \vdash A$ and $T \vdash \neg A$ for Boolean A. We remind the reader that for $i \neq j$ the formulas $(i)A$, $(j)A$ are incompatible; thus for given A, a consistent theory T may contain at most one formula of the form $(i)A$. If T is consistent and complete, then for each A, there is exactly one i such that $(i)A \in T$.*

A theory T is maximal consistent if T is consistent and all theories $T' \supset T$ are inconsistent. T is complete if for each Boolean A, $T \vdash A$ or $T \vdash \neg A$. T is closed under provability if for each Boolean A we have that $T \vdash A$ implies that $A \in T$.

LEMMA 5.2 *Each consistent theory T has a maximal consistent extension T'.*

Proof By the usual Lindenbaum construction. The novice may consult any standard textbook of logic, e.g. [15]. ■

LEMMA 5.3 *Maximal consistent theories are complete and closed under provability.*

Proof As usual, remembering that for Boolean theories we have full propositional logic, including the deduction theorem; thus if $T \nvdash A$, then $(T \vdash \neg A)$ is consistent. ■

DEFINITION 5.2 *Let T be a maximal consistent theory; for each formula A, put $e(A) = i$ iff T contains the formula $(i)A$. An* evaluation *is an e given by a maximal consistent theory T.*

LEMMA 5.4 *If e is as above, then e commutes with connectives, i.e., $e(A \wedge B) = \min(e(A), e(B))$, etc.; furthermore, $e(A) = 1$ for each* MTL*-provable A.*

Proof We have $e(A \wedge B) = i$ iff $(i)(A \wedge B) \in T$; also the following formula is in T: $(j)A \wedge (k)B \rightarrow (\min(j, k))(A \wedge B)$. Take j, k such that $j = e(A)$, $k = e(B)$; then $(j)A \wedge (k)B \in T$, hence $(\min(j, k))(A \wedge B) \in T$, thus $i = \min(j, k)$. ■

LEMMA 5.5 *Let e, e' be evaluations. Then the following are equivalent:*

(1) *for each A, $e(GA) \leq e'(A)$;*

(2) *for each A, $e(A) \leq e'(PA)$;*

(3) *for each* A, $e(A) \geq e'(HA)$;

(4) *for each* A, $e(FA) \geq e'(A)$.

Proof Assume (1) and prove (2): since $A \to GPA$ is an axiom, $e(A) \leq e(GPA) \leq e'(PA)$.

Assume (4) and prove (1): Suppose $e(GA) = i$. Then $e(F(\neg A)) = e(\neg GA) = 1 - i$. By (4), $e(F(\neg A)) \geq e'(\neg A)$, so $e'(\neg A) \leq 1 - i$ and $e'(A) \geq i$.

The rest is similar. ■

DEFINITION 5.3 *For evaluations* e, e', *let* $R(e, e')$ *mean that for each* A, $e(GA) \leq e'(A)$.

LEMMA 5.6 *R is reflexive, transitive, not branching towards the past, and not branching towards the future.*

Proof Reflexivity: We have to show that for all formulas A and for all evaluations e, $e(GA) \leq e(A)$. Suppose that e is given by the theory T, and suppose that $e(GA) = i$, i.e. $(i)GA \in T$. We know that also $(1)(GA \to A) \in T$, so by (11), $(\geq i)A \in T$, i.e. $e(A) \geq i$.

Transitivity: Suppose $e_0 R e_1$ and $e_1 R e_2$, where e_0, e_1, e_2 are given by T_0, T_1, T_2 respectively. We have to prove that for all A, $e_0(GA) \leq e_2(A)$. Suppose $e_0(GA) = i$, i.e. $(i)GA \in T_0$. We know that $(1)(GA \to GGA) \in T_0$, so by (11), $(\geq i)GGA \in T_0$, i.e. $e_0(GGA) \geq i$. But then $e_0 R e_1$ gives $e_1(GA) \geq i$, and $e_1 R e_2$ gives $e_2(A) \geq i$, as desired.

Not branching towards the future: Suppose $e_0 R e_1$ and $e_0 R e_2$, where e_0, e_1, e_2 are given by T_0, T_1, T_2 respectively. We have to prove that $e_1 R e_2$ or $e_2 R e_1$. In order to derive a contradiction, suppose that neither disjunct holds. Then there are A, B such that $e_1(A) > e_2(PA)$ and $e_1(B) > e_2(FB)$. Suppose $e_1(A) = a$, $e_1(B) = b$.

Then $T_1 \vdash (\geq a)A \wedge (\geq b)B$, whereas $T_2 \vdash (< a)PA \wedge (< b)FB$. Because $e_0 R e_1$, we find $T_0 \vdash F((\geq a)A \wedge (\geq b)B)$, so by axiom (6),

$$T_0 \vdash G(P((\geq a)A \wedge (\geq b)B) \vee F((\geq a)A \wedge (\geq b)B)).$$

Because $e_0 R e_2$, this implies

$$T_2 \vdash P((\geq a)A \wedge (\geq b)B) \vee F((\geq a)A \wedge (\geq b)B);$$

in particular, by (13), $T_2 \vdash (\geq a)PA \vee (\geq b)FB$, contradicting our assumption. The proof that R is not branching towards the past is analogous. ■

LEMMA 5.7 *For every Boolean formula A and consistent theory $T \supseteq \{FA\}$, there is a consistent theory $T' \supseteq \{A\}$.*

Proof Let T' be just A plus the set Mtl of all MTL-provable Boolean formulas. If T' is inconsistent, then Mtl $\vdash \neg A$, so $\neg A \in$ Mtl; thus $G(\neg A) \in$ Mtl and T is inconsistent. ■

LEMMA 5.8 (Valuation lemma) *For all evaluations e and formulas A, $e(FA) = \max_{e R e'} e'(A)$. Similarly for the other modal operators.*

Proof Suppose $e(FA) = i$. First, (4) of Lemma 5.5 immediately implies that for all e' with $e R e'$, $e'(A) \leq i$. It remains to show that there is an e' such that $e R e'$ and $e'(A) = i$.

We show that the theory $T' = \mathrm{Mtl} \cup \{(i)A\} \cup \{(\geq j_B)B \mid e(GB) = j_B\}$ is consistent. Any maximal consistent extension of T' then defines an appropriate e'. Take $B_1, \ldots, B_k$ and let $j_m = e(GB_m)$. Let T be the theory defining e. We have

$$T \vdash (i)FA \wedge \bigwedge_m (j_m)GB_m;$$

thus

$$T \vdash F\left((i)A \wedge \bigwedge_m (\geq j_m)B_m\right);$$

thus by Lemma 5.7, $\mathrm{Mtl} \cup \{(i)A\} \wedge \{(\geq j_m)B_m \mid m\}$ is consistent. By compactness, T' itself is consistent. This finishes the proof. ■

THEOREM 5.1 (Completeness theorem) MTL $\vdash A$ *iff A has the value* 1 *in all worlds of all Kripke models with reflexive linearly preordered time.*

Proof $\Rightarrow$: By Lemma 3.4.

$\Leftarrow$: If A is not provable, then $(1)A$ is not provable; thus $(< 1)A$ is a consistent Boolean formula, so there is an evaluation e_0 with $e_0(A) < 1$.

Define the model $K = \langle W, \Vdash, \leq \rangle$, where W is the set of all evaluations e such that $e_0 R e$ or $e R e_0$, where R is as in Definition 5.8, $\leq$ is R restricted to W, and $\Vdash$ is defined by $\|p\|_e = e(p)$ for propositional variables p.

Now it is easy to check by induction on the complexity of the formula, using Lemma 5.4 and the valuation Lemma 5.8, that for all e, B one has $\|B\|_e = e(B)$ [we leave the (i) step to the reader]. In particular, $\|A\|_{e_0} < 1$. Moreover, by Lemma 5.6 we conclude that $\leq$ is a reflexive linear preorder on W. This finishes the proof. ■

In order to prove Lemma 3.2, the following lemma suffices:

LEMMA 5.9 (Finite-model property) *If $\langle W, \Vdash, \leq \rangle$ is a reflexive linearly preordered Kripke model such that for some $w_0 \in W$ one has $\|A\|_{w_0} < 1$, then there is a* finite *reflexive linearly preordered Kripke model $\langle W_{\mathrm{fin}}, \Vdash_{\mathrm{fin}}, \leq_{\mathrm{fin}} \rangle$ and a $w' \in W_{\mathrm{fin}}$ such that $\|A\|^{\mathrm{fin}}_{w'} < 1$.*

Proof Let Φ be the closure of $\{A\}$ under subformulas and single negations. Define for $w_1, w_2 \in W$

$$w_1 \sim w_2 \quad \text{iff} \quad \text{for all } B \in \Phi, \quad \|B\|_{w_1} = \|B\|_{w_2}.$$

Define $W_{\text{fin}} = \{[w] \mid w \in W\}$, the finite set of equivalence classes.

Define $[w_1] \leq^* [w_2]$ iff $w_1' \leq w_2'$ for some $w_1' \in [w_1]$, $w_2' \in [w_2]$. Let $\leq_{\text{fin}}$ be the transitive closure of $\leq^*$. It is clear that $\leq_{\text{fin}}$ is reflexive and connected by inheritance from $\leq$, so it is a reflexive linear preorder.

Finally, define $\|p\|^{\text{fin}}_{[w]} = \|p\|_w$ for all propositional variables $p \in \Phi$, $[w] \in W_{\text{fin}}$, and define $\Vdash_{\text{fin}}$ arbitrarily for $p \notin \Phi$. Now one can straightforwardly check by induction on the complexity of the formula that for all $B \in \Phi$, $[w] \in W_{\text{fin}}$, one has $\|B\|^{\text{fin}}_{[w]} = \|B\|_w$. In particular $\|A\|^{\text{fin}}_{[w_0]} < 1$. This finishes the proof. ■

Remark 5.1

(1) We have shown in the main theorem that both QFL and QFL′ embed faithfully to MTL and presented a complete axiom system for MTL. Thus a formula Φ of QFL is a 1-tautology with respect to the semantics of QFL iff its translation Φ^* is provable in MTL. Recall that the only modality of QFL is $\lhd$ (binary), whereas the modalities of MTL are G, H (unary). The problem remains to find an elegant axiomatization of 1-tautologies of QFL in the language of QFL. (A pedestrian axiomatization is easy to obtain by reducing everything to Boolean formulas [in particular reducing $A \lhd B$ to a formula involving only $(i)A \lhd (j)B$] and applying the axiom system of [1].) In attacking this problem one may start by testing axioms of QPL: some of them are 1-tautologies of our logic (e.g. transitivity), but some are not [e.g. dichotomy $(A \lhd B) \vee (B \lhd A)$]. For example, is there a complete axiom system for QFL (and hence for QFL′) whose axioms concerning modalities do not contain coefficients?

(2) Moreover, one can investigate logics based on comparison of other fuzzy truth values, infinitely valued systems, and many other variations. The purpose of the present paper is mainly to show the direction for future research.

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